Verification of the trajectories of Stewart platform manipulators against singularities

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Abstract

This paper deals with the problem of determining conclusively if a given spatial trajectory of a semi-regular Stewart platform manipulator (SRSPM) passes through singularities. The algorithm presented here is restricted to trajectories defined in terms of polynomial functions of a continuous path parameter. As such it can complement existing motion-planning strategies which use piecewise continuous cubics or other polynomials to describe the trajectories. The algorithm, however, can be extended to functions such as rational or trigonometric, which can be reduced readily to polynomial forms.

The advantage of the formulation presented here is that the computation is relatively simple and the results are guaranteed theoretically. Further, if a given trajectory does pass through a singularity, the algorithm returns the set of points where it does so. The theory presented in the paper is illustrated with numerical examples involving 3-D and 6-D trajectories defined in terms of cubic polynomials in a single path parameter.
Keywords: Stewart platform, Singularity, Trajectory verification, Cubic spline and Polynomials

1 Introduction

Parallel manipulators, such as the Stewart platform, are becoming increasingly popular due to their demonstrated advantages over the serial counterparts in terms of load capacity, rigidity, accuracy and speed. However, one of the inherent problems in such manipulators is the existence of singularities within their workspaces. In general, for a parallel manipulator, it is difficult to find a description of the singularity manifold which is easily amenable to analysis. Consequently, the task of planning of a desired path or trajectory which is provably free of singularities is fairly hard.

In this paper, we take a step closer to the final objective stated above. We propose a formulation which can test a given trajectory, *with guarantee*, for singularities. In case the trajectory does have singular poses of the manipulator within it, these are obtained assuredly. The theory presented here builds on the results on the singularity manifold of an SRSPM presented in [1]. The present form of the algorithm has the following restrictions:

- The algorithm depends on closed-form expressions describing the singularity manifold. To the best of the author's knowledge, such expressions have been derived only for the case of a SRSPM in which both the fixed and moving platforms are hexagonal in shape, and in each platform, the alternate sides have equal lengths. Therefore, the scope of the present work is restricted to that manipulator. However, it is worth noting at this point that most of the Stewart platform manipulators used in research or industries do fall in this category.

- It is assumed that the desired trajectory is defined in terms of polynomial expressions of a single path parameter. This restriction is crucial to the formulation, as the "guaranteed" nature of the results derives from the fact that the problem of the intersection of a given trajectory with the singularity manifold is reduced to solving a univariate polynomial equation. However, this requirement is not very restrictive as there are existing motion planning algorithms based on cubic splines (see, e.g. [2] and the references therein) and the present algorithm can complement these naturally. Further, rational and trigonometric equations can be converted into polynomial equations with some manipulation, and therefore trajectories described in terms of these can also be validated in the above formulation, albeit with some additional effort in pre-processing.

- Potentially, the most severe drawback of the proposed algorithm is that it requires a robust numerical tool for solving univariate polynomial equations. The final equation to solve can be easily of moderate to very high degree (e.g. of degree 27 if all the six pose parameters are described in terms of cubic functions of the path parameter). While such solvers are readily available in most commercial packages or numerical codes, one still needs to be careful in checking the numerical accuracy of the end-results before trusting them.

The paper is organised as follows: in section 2, the detailed theoretical formulation is presented. In section 3, a few numerical examples are presented. Finally, the conclusions are presented in section 4.

2 Formulation

The algorithm proposed in this paper can be discussed in two parts:

1. Approximation of the given trajectory in terms of piece-wise continuous polynomial functions of the path parameter.

2. Exact verification each segment of the above trajectory for singularity.

The focus of the paper is on the second part, where it is assumed that such a description of the trajectory is available to that part of the algorithm. However, for the sake of completeness, we describe below why such a description is needed, and indeed, it can be constructed with reasonable effort based on methods established in literature.

2.1 Approximation of input trajectory

In general, the SRSPM can execute a trajectory in which all the configuration variables, expressed as functions of time, can show substantial variety and complexity. However, in this work, we will start by imposing some structure to the actual trajectories, and that we will do by approximating the desired trajectory using a model, cubic-splines in this case. This step is necessary for two reasons:

- Firstly, in many applications, the desired trajectory of an SRSPM is fairly complicated, and difficult to be commanded directly to the manipulator. For instance, these manipulators are widely used as motion simulators for aircrafts and ground vehicles, wherein the desired trajectories are essentially the numerically computed responses of the mathematical models representing such mechanical systems. Therefore, such trajectories are defined only at discrete points, and require to be approximated by continuous functions before they can be commanded to the actual manipulator.

- Secondly, even if the desired trajectories be available as a set of continuous functions of time, it is difficult in general, and indeed impossible in certain cases, to validate them conclusively for singularities. The singularity conditions are available as trigonometric and/or polynomial equations [3, 4, 5, 6, 1]. In order for an exact check against singularities, the commanded trajectories when incorporated into the singularity conditions should generate a system of equations, of which all roots can be computed, at least numerically. If, for instance, there is an exponential term in any of the trajectories, then the resulting set of polynomial/trigonometric and exponential equations *cannot be solved* even numerically to obtain *all the solutions* in general. The only forms of functions compatible with the existing conditions for singularity are the trigonometric, polynomial and rational functions. We will adopt the polynomial form in this work, since our description of singularity following [1] is polynomial in nature.

2.2 Verification of the trajectory

In the following, we describe how the trajectory developed above can be checked for singularity using our algorithm. Since the present focus is on the SRSPM, a description of the manipulator may be in order.

2.2.1 Geometry of the SRSPM

As mentioned earlier, the SRSPM has hexagonal top and bottom platforms, with alternate sides in each platform having identical length. There is a 3-way symmetry in each platform, and the adjacent pairs of legs are arranged symmetrically about the three radial lines of symmetry in each platform. The angular spacings between the adjacent pairs of legs are denoted by $2\gamma_t$, and $2\gamma_b$ for the top and bottom platforms respectively. The manipulator along with the frames of reference used is shown in figure 1(a), and the bottom platform, in figure 1(b).

Without any loss of generality, the radius of the circum-circle of the bottom platform is scaled to unity[1] and thereby one architectural parameter is eliminated from all subsequent calculations. Radius of the circum-circle of the top platform is denoted by r_t.

The centre of the top platform is described in the base frame as $p = (x,y,z)^T \in \mathbb{R}^3$. Its orientation is described by the rotation matrix $R \in SO(3)$ with respect to the base frame.

2.2.2 Mathematical development

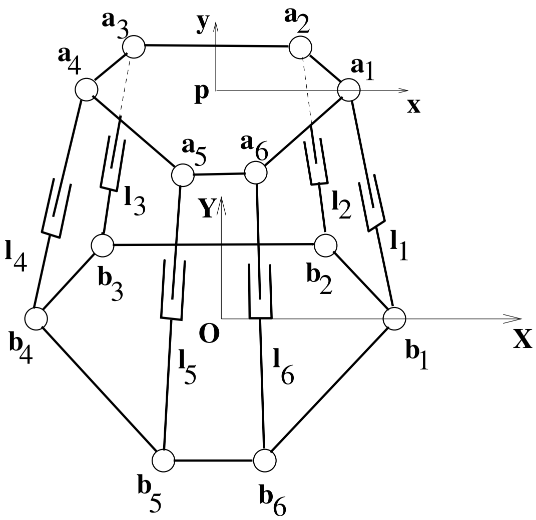
Consider the motion of the SRSPM such that p moves from p_1 to p_2 in time $t \in [0,T]$. Simultaneously, R changes from R_1 to R_2. It is obvious that the path[2] traced by the platform can be parametrised by $s \in [0,1]$ such that $p(s=0) = p_1$ and $p(s=1) = p_2$ and $p(s)$ gives the position of the centre of the top platform at any instance and so on. It is fairly common in robotics literature to approximately fit such a path in terms of piece-wise continuous cubic polynomials (see, e.g. [7]). Obviously, a similar description can be achieved through cubic-splines.

For approximating the orientation, once again a cubic spline-based method is followed. Kang and Park [2] presents orientation interpolation schemes in terms of cubic splines. In this paper, we interpolate orientation in terms of cubic splines in the domain of Rodrigue's parameters. Suppose, the desired orientation is specified at the points $s = 0, s_1, s_2, \ldots, s_{n-2}, s_{n-1}, 1$, in terms of rotation matrices $R_i = R(s_i) \in SO(3)$. The matrix R_i can be reduced to the *axis-angle* form in terms of the rotation axis $u \in \mathbb{R}^3$, $\|u\| = 1$ and rotation angle $\theta \in [0,\pi]$ (see, e.g. [7]). From these it is easy to calculate the corresponding Rodrigue's parameters:

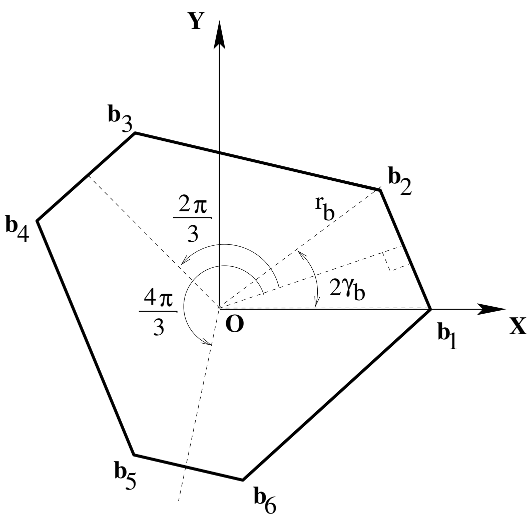
$$c = (c_1, c_2, c_3)^T = u \tan(\theta/2) \tag{1}$$

[1] We use *radians* for the angular unit, while the length unit for base platform can be chosen as convenient. All other length entities appearing in the paper are non-dimensionalised accordingly.

[2] In this context, the term *path* is used to include the variation in orientation as well as the position of the top platform.



(a) The manipulator



(b) Bottom platform

Figure 1: Geometry of the Stewart platform manipulator

Once the parameters c are calculated at the nodal points, they can be interpolated through a cubic formulation, such that the resulting continuous description of orientation in terms of the path parameter is given by:

$$c_j(s) = c_{j0}s^3 + c_{j1}s^2 + c_{j2}s + c_{j3}, \quad s \in [s_i, s_{i+1}],\ j = 1,2,3 \tag{2}$$

where the subscript i serves to indicate the ith interval in the parameter space. A similar formula can be obtained easily for the position, $p(s) = (x(s), y(s), z(s))^T$, i.e.

$$\begin{aligned} x(s) &= x_0s^3 + x_1s^2 + x_2s + x_3 \\ y(s) &= y_0s^3 + y_1s^2 + y_2s + y_3 \\ z(s) &= z_0s^3 + z_1s^2 + z_2s + z_3,\ s \in [s_i, s_{i+1}] \end{aligned} \tag{3}$$

2.3 Validation of the trajectory

Once the configuration parameters, $x(s), y(s), z(s)$ and $c_1(s), c_2(s), c_3(s)$ are obtained as above, they are inserted in the singularity condition. As noted in [1], the condition is given by:

$$\begin{aligned} &E_1x^2z + E_2x^2 + E_3xyz + E_4xy + E_5xz^2 + E_6xz + E_7x \\ &+ E_8y^2z + E_9y^2 + E_{10}yz^2 + E_{11}yz + E_{12}y + E_{13}z^3 \\ &+ E_{14}z^2 + E_{15}z + E_{16} = 0 \end{aligned} \tag{4}$$

where $E_i,\ i = 1, \ldots, 16$, are functions of the architecture variables of the SRSPM and its orientation parameters c. In particular, c_1, c_2, c_3 appear in E_i's only as polynomials.

The expressions for E_i are too big to be included here. However, two of them are quoted below as illustrations [1]:

$$\begin{aligned} E_{14} =& 8({c_1}^2 + {c_2}^2 - {c_3}^2 - 1)r_t \sin(\gamma + 3\gamma_t)((c_3{c_1}^3 \\ & - 3c_2{c_1}^2 - 3{c_2}^2c_3c_1 + {c_2}^3)\cos(\gamma + 3\gamma_t) \\ & + ({c_1}^3 + 3c_2c_3{c_1}^2 - 3{c_2}^2c_1 - {c_2}^3c_3)\sin(\gamma + 3\gamma_t)) \\ & \times \sin^2(\gamma) + 8({c_1}^2 + {c_2}^2 + {c_3}^2 + 1) \times \sin(2\gamma + 3\gamma_t) \\ & \times ((c_3{c_1}^3 + 3c_2{c_1}^2 - 3{c_2}^2c_3c_1 - {c_2}^3)\cos(2\gamma + 3\gamma_t) \\ & - ({c_1}^3 - 3c_2c_3{c_1}^2 - 3{c_2}^2c_1 + {c_2}^3c_3)\sin(2\gamma + 3\gamma_t)) \\ & \times \sin^2(\gamma) \end{aligned} \tag{5}$$

$$\begin{aligned} E_{15} =& -8({c_1}^2 + {c_2}^2)({c_1}^2 + {c_2}^2 - {c_3}^2 - 1){r_t}^2 \sin(\gamma) \\ & \times (({c_3}^2 - 1)\cos(\gamma) - 2c_3\sin(\gamma))\sin^2(\gamma + 3\gamma_t) \\ & + 8({c_1}^2 + {c_2}^2)({c_1}^2 + {c_2}^2 + {c_3}^2 + 1)\sin(\gamma) \\ & \times (({c_3}^2 - 1)\cos(\gamma) - 2c_3\sin(\gamma))\sin^2(2\gamma + 3\gamma_t) \\ & + ({c_1}^2 + {c_2}^2)r_t(-4{c_3}^3 + 4({c_3}^2 - 1)\cos(4\gamma)c_3 \\ & + 4c_3 - 2({c_3}^2 + 1)^2\sin(2\gamma) \\ & + 4\cos(6\gamma_t)(({c_3}^2 - 1)\cos(\gamma) \\ & - 2c_3\sin(\gamma))^2(\sin(2\gamma) - \sin(4\gamma)) + (3{c_3}^4 - 2{c_3}^2 + 3) \\ & \times \sin(4\gamma) + 8(2\cos(2\gamma) + 1)\sin^2(\gamma)(-\cos(\gamma){c_3}^2 \\ & + 2\sin(\gamma)c_3 + \cos(\gamma))^2\sin(6\gamma_t)) \end{aligned}$$

where $\gamma = \gamma_b - \gamma_t$. As can be seen in equation (4), equation (5) respectively, the singularity condition is of degree 3 in x, y, z and degree 6 in c_1, c_2, c_3. It may be noted here that though all the E_i's have not been presented here, E_{14}, E_{15} are indeed representative in this regard. Therefore, when equations (2, 3) are used in equation 4, we obtain a univariate polynomial in the path parameter s, whose degree, in the general case is given by $3 \times 3 + 3 \times 6 = 27$.

In principle, all the roots of the polynomial equation can be obtained, albeit numerically. If there are real roots in the interval under consideration, i.e. $s \in [s_i, s_{i+1}]$, then the trajectory meets with singularity in that interval. Otherwise, the whole interval is free of singularities[3].

3 Illustrative examples

In this section, we demonstrate the formulation presented above through a few numerical examples. We adopt the architectural parameters from the INRIA manipulator as described in [1]:

$$r_t = 0.5803,\ \gamma_b = 0.2985 \text{ rad},\ \gamma_t = 0.6573 \text{ rad}.$$

For the sake of simplicity, we consider the motion interval as $s \in [0, 1]$ and note that this does not cause any loss of generality.

3.1 Example 1: motion with constant orientation

In this case, we assume the constant orientation to be given by the Rodrigue's parameters:

$$c_1 = 0,\ c_2 = 0.1,\ c_3 = 0.1$$

The chosen end points are:

$$p_1 = (-0.5, -0.5, 0.5)^T,\ p_2 = (0.5, 0.5, 1)^T$$

The desired path is a straight line from p_1 to p_2. The motion is taken to start and stop with zero velocity, leading to the following parameterisation of the path:

$$\begin{aligned} x(s) &= -2s^3 + 3s^2 - \frac{1}{2} \\ y(s) &= -2s^3 + 3s^2 - \frac{1}{2} \\ z(s) &= \frac{1}{2}\left(3s^2 - 2s^3\right) + \frac{1}{2},\ s \in [0, 1] \end{aligned}$$

The numerical form of the univariate polynomial equation in s is given by[4]:

$$\begin{aligned} &0.482678s^9 - 2.17205s^8 + 3.25808s^7 - 2.1713s^6 \\ &+ 1.62679s^5 - 1.22009s^4 + 0.20512s^3 \\ &- 0.30768s^2 - 0.0263167 = 0 \end{aligned} \quad (6)$$

[3]It is possible to obtain a real root for s outside the interval concerned. However, that is not of any consequence as the equation (4) is valid only in one interval at a time.

[4]All the numerical values in this paper are correct up to the seventh place after the decimal point. However, for the sake of brevity, the trailing zeros have not been included.

It turns out that the only real solution of the polynomial is $s = 1.65682$, which is outside the relevant range of s. Therefore it can be stated conclusively that the above trajectory is free of singularity.

3.2 Example 2: motion with constant position

In this example, we fix the position at:

$$p(s) = p_1 = (-0.5, -0.5, 0.5)^T,\ s \in [0, 1]$$

The orientation parameters change from

$$c_1 = (0, 0.1, 0.1)^T$$

to

$$c_2 = (6, 3.1, 6.1)^T$$

The desired orientation trajectory is taken to consist of arbitrary cubic functions of s, except that the end conditions are satisfied:

$$\begin{aligned} c_1(s) &= s^3 + 2s^2 + 3s \\ c_2(s) &= s^3 + s^2 + s + 0.1 \\ c_3(s) &= 3s^3 + 2s^2 + s + 0.1,\ s \in [0, 1] \end{aligned}$$

As expected, this results in a polynomial of degree 18, given below:

$$\begin{aligned} &-286.55s^{18} - 1541.52s^{17} - 4642.01s^{16} - 8977.38s^{15} \\ &-11840.9s^{14} - 10137.s^{13} - 4403.48s^{12} + 1483.37s^{11} \\ &+4684.03s^{10} + 4522.8s^9 + 2564.23s^8 + 723.751s^7 \\ &-75.2902s^6 - 128.399s^5 - 65.2046s^4 - 22.994s^3 \\ &+10.7222s^2 + 0.042965s - 0.0263167 = 0 \end{aligned} \quad (7)$$

In this case, there are a number of real roots in $[0, 1]$:

$$s = 0.0507524,\ 0.250645,\ 0.346431,\ 0.786169$$

Therefore it is clear that the trajectory passes through singular configurations corresponding to the four values of s given above.

3.3 Example 3: general spatial motion

In this case both the above motions take place simultaneously, i.e. $p(s)$ varies from $p_1(s)$ to $p_2(s)$, and $c(s)$ varies between $c_1(s)$ and $c_2(s)$ along the trajectories defined above. The singularity condition reduces to a degree 27 equation in s in this case:

$$\begin{aligned} &-2324.34s^{27} - 1649.52s^{26} + 2304.26s^{25} + 16225.6s^{24} \\ &+18471.6s^{23} + 3630.12s^{22} - 34359.2s^{21} - 55269.5s^{20} \\ &-33314.7s^{19} + 38913.3s^{18} + 96743.9s^{17} + 70623.9s^{16} \\ &-44537.7s^{15} - 172359.s^{14} - 225083.s^{13} - 200538.0s^{12} \\ &-133203.s^{11} - 66127.1s^{10} - 20385.7s^9 - 779.352s^8 \\ &+2502.97s^7 + 929.977s^6 + 39.8743s^5 - 45.4224s^4 \\ &-22.1957s^3 + 10.4146s^2 + 0.042965s - 0.0263167 = 0 \end{aligned} \quad (8)$$

The trajectory is seen to encounter singularities at $s = 0.0512808$, 0.302583.

4 Conclusion

It has been shown in this paper how the task of validation of the spatial trajectories of an SRSPM can be reduced to the finding the real roots of a univariate polynomial within a given range. If the desired trajectories, approximated by cubic polynomials in terms of a path parameter are actually commanded to the manipulator, then the validation results are *guaranteed*, provided the numerical coefficients and the roots of the univariate polynomial are calculated reliably. Several numerical examples have been provided to illustrate the formulation developed in the paper.

To the best of the author's knowledge, the present method for trajectory validation is novel. It provides an improvement over the methods of grid-based checking, of which the results are not guaranteed. It is hoped that the present work would help researchers in the process of developing singularity-free trajectories for SRSPM's and similar manipulators.

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