Dynamic analysis of Cantilever beam with transverse crack

Isham Panigrahi 1[1*] and Dayal Ramakrushna Parhi 2[1]
[1]Mechanical Engineering Department,KIIT University,Bhubaneswar,India
[2] Mechanical Engineeing Department,NIT Rourkela,Rourkela,India

[*] Corresponding author (email: i_panigrahi@yahoo.com)

Abstract

This paper contains an attempt to evaluate dynamic behaviors of beam structures with transverse crack subjected to external force. In this work theoretical expressions have developed for finding out the mode shapes and natural frequencies for beam with transverse crack using flexibility influence coefficients and local stiffness matrix. Crack depth and crack position are taken as main variable parameters. Suitable numerical models are considered, and the results are presented graphically. Further experimental and finite element analysis verifications are also done to prove the authenticity of the theory developed. The work leads to the conclusion that, the presence of crack in structure makes an appreciable difference in dynamic response.

Keywords: Crack, Beam, FEM, and Vibration

Nomenclature

a_1 =depth of crack, mm

A =cross-sectional area of the beam , mm^2

A_i ,i=1 to 12=co-efficients of matrix A

B =width of the beam, mm

C_{ij} = flexibility influence co-efficient

E =young's modulus of beam material, N/mm^2

F_i ,i=1,2=experimentally determined function

J =strain-energy release rate

K_{Ii} ,i=1,2=stress intensity factors for P_i loads

K_{ij} =local flexibility matrix elements

L =length of the beam, mm

L_1 =location of the crack from fixed end , mm

P_i ,i=1,2=axial force(i=1),bending moment(i=2)

Q =stiffness matrix for free vibration

u_i ,i=1,2=normal functions longitudinal

y_i ,i=1,2=normal functions transverse

W =depth of the beam, mm

ω =natural circular frequency, rad/s

β =relative crack location($\frac{L_1}{L}$)

$\mu = A\rho$, kg/mm

ρ =mass density of beam, kg/mm^3

ξ_1 =relative crack depth ($\frac{a_1}{W}$)

1 Introduction

The development of high speed machineries and light weight rising structures, fault diagnosis using the behaviors of different components of the system has gained paramount importance. It has also been realized that the presence of crack in structures or in machine member leads to operational problems as well as premature failure. Major characteristics of the structure which undergo change due to presence of crack are the natural frequency, the amplitude response due to vibration and mode shape. Scientific study on the changes in these characteristics can be widely utilized for the identification of crack in structures.

In this investigation, the presence of transverse crack in the structure has been considered. This crack introduces new boundary conditions for the structure at the crack location. These boundary conditions are derived from the strain energy equation using Castigliano's theorem. Presence of crack also reduces the stiffness of the structure which has been derived from the stiffness matrix. For dynamic behaviors of beam with a transverse crack, Timoshenko beam theory with modified boundary conditions have been used to find out the theoretical expressions for the natural frequencies and the modes for the beam. For all the theoretical expressions as derived for dynamic characteristics of structure with a crack, respective numerical analysis was taken up with suitable numerical models with the help of the computer.

In order to establish the authenticity of theories developed, experiments, finite element analysis have been conducted in varied specimens in line with the numerical models adopted in different sections. Experimental, finite elements and analytical results have been compared and are showing good agreement. From the present investigations, the following generalized results are achieved as expected. Presence of crack in structure makes an appreciable difference in dynamic response. These findings can be utilized in various industrial applications, particularly for fault detection on structures using condition monitoring technique.

2 Background of the Analysis

A local compliance has been used to quantify, in a microscopic way, the relation between the applied load and the strain energy concentration around the tip of crack by Irwin [1, 2]. This idea has been implemented for determining stress intensity factor, describing the intensity of the stress field about the tip of the crack. This becomes a standard method for calculating the stress intensity factors and both analytical and experimental results are tabulated for number of cases, different in loading and geometry conditions [3]. A general method has been considered by Okamura et al. [4] for applying fracture mechanics through the local compliance concept for the analysis of a structure containing cracked members. Krawczuk and Ostachowicz [5, 6, 7] have analyzed the effect of positions and depths of two cracks on the natural frequencies of a cantilever beam. . Expressions for bending vibrations of an Euler-Bernoulli cracked beam have been analysed by Matvev etal. [8]. They have studied the effects of the ratio of crack location to the length of the beam and also the ratio of depth of the crack to the height of the beam.They have investigated the variation of the natural frequency of the cracked beam.Chondros et al. [9] have analysed the lateral vibration of cracked Euler-Bernoulli beams with single or double edge cracks. Their analysis can be used for the prediction of the dynamic response of a simply supported beam with open surface cracks. Qian et al. [10] have used a finite element model to analyze the effect of crack closure on the transverse vibration of a beam. The stiffness matrix of the system has been deduced from the stress intensity factors, and it gives two values, one for the close crack (uncracked beam) and for the other for the open crack. The sign of the stress on the crack faces has been used to determine if the crack is open or closed at each time step.Fernandez-Saez et al. [11] have used the method of flexibility influence functions to approximate the fundamental frequency for bending vibrations of cracked Euler Bernoulli beams with different boundary conditions. The results obtained agree with those numerically obtained by the finite element method. A simple method for predicting the locations and depths of the cracks based on changes in the natural frequencies and amplitudes of the frequency response functions (FRFs) of the beam has also been presented and discussed. Maiti [12]. A method to measure a change in crack length from the change in the first natural frequency has been presented. The accuracy of the methods is illustrated by case studies involving a short cantilever beam with a crack. Both numerical and experimental case studies are presented to demonstrate the effectiveness of the methods. Zheng and Kessissoglou [13] have used the overall additional flexibility matrix instead of the local additional flexibility matrix to obtain the total flexibility matrix of a cracked beam. The stiffness matrix is then obtained from the total flexibility matrix and is used for calculating the natural frequencies of a cracked beam.Narkis and Elmalah [14] have demonstrated the possibility of crack identification in cantilever beams under uncertain end conditions, using natural frequency variation. They have developed a method for characterizing the effect of clamp rigidity on free vibrations of the beam, and for direct calculation of crack location based on variations of three natural frequencies. The proposed method has been validated both numerically and experimentally. Dharamraju et al. [15] have developed an algorithm for estimation of beam crack parameters. The analysis is based on the finite element method. The identification methods rely on the measurement of the beam response for a known sinusoidal force when the uncracked beam model and the crack location are known. The identification algorithm is illustrated by a simulated example. Kim and Stubbs [16] have presented a methodology to non-destructively locate and estimate size of crack in structures for which only a few natural frequencies are available. The proposed methodology is presented in two parts. The first part of the paper has outlined a theory of crack detection that yield information on the location and size of crack directly from changes in frequencies of the structures. The second part of the paper has demonstrated the feasibility and practicality of the crack detection scheme by accurately locating and sizing cracks in test beams.

3 Theoretical Analysis

For theoretical analysis a cantilever beam of width 'B', height 'W' and length 'L' with a transverse crack of depth 'a_1'for full width at a distance 'L_1' from fixed end of the beam is taken.

3.1 Local Flexibility of a Cracked Beam under Bending and Axial Loading

The presence of a transverse surface crack on a beam introduces a local flexibility, which can be defined in matrix form, the dimension of which depends on the degrees of freedom .Here a 2x2 matrix is considered .A cantilever beam is subjected to axial force P_1 and the bending moment P_2 which gives coupling with the longitudinal and transverse motion.

As per Tada, the strain energy release rate at the fractured section can be written as,

$$J = \frac{1}{E'}(K_{I1} + K_{I2})^2$$

$$\frac{1}{E'} = \frac{1-\nu^2}{E} \text{ (for plane strain condition)}$$

$$=\frac{1}{E} \quad \text{(for plane stress condition)}$$

Where,

K_{I1}, K_{I2} are the stress intensity factors of mode I (opening of the crack) for load P_1 and P_2 respectively. The values of stress intensity factors from earlier studies are,

$$K_{I1} = \frac{P_1}{BW}\sqrt{\pi a}(F_1(\frac{a}{W}))$$

$$K_{I2} = \frac{6P_2}{BW^2}\sqrt{\pi a}(F_2(\frac{a}{W}))$$

Where expressions for F_1 and F_2 are as follows

$$F_1(\frac{a}{W}) = (\frac{2W}{\pi a}\tan(\frac{\pi a}{2W}))^{0.5}$$

$$\left\{\frac{0.752 + 2.02(a/W + 0.37(1-\sin(\pi a/2W)^3)}{\cos(\pi a/2W)}\right\}$$

$$F_2\left(\frac{a}{W}\right) = \left(\frac{2W}{\pi a}\tan\left(\frac{\pi a}{2W}\right)\right)^{0.5}$$

$$\left\{\frac{0.923 + 0.199(1-\sin(\pi a/2W))^4}{\cos(\pi a/2W)}\right\}$$

Let U_t be the strain energy due to the crack. Then from Castigliano's theorem, the additional displacement along the force P_i is

$$U_i = \frac{\partial U_t}{\partial P_i} \tag{1}$$

The strain energy will have the form

$$U_t = \int_0^{a_1} \frac{\partial U_t}{\partial a} da = \int_0^{a_1} J da \tag{2}$$

Where, $J = \dfrac{\partial U_t}{\partial a}$ is the strain energy density function from (1) and (2) thus we have

$$U_i = \frac{\partial}{\partial P_i}\left[\int_0^{a_1} J(a)da\right] \tag{3}$$

The flexibility influence co-efficient C_{ij} will be, by definition

$$C_{ij} = \frac{\partial U_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j}\int_0^{a_1} J(a)da \tag{4}$$

To find out the final flexibility matrix we have to integrate over the breadth 'B'.

$$C_{ij} = \frac{\partial U_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j}\int_{-B/2}^{+B/2}\int_0^{a_1} J(a)dadz \tag{5}$$

Putting the value of strain energy release rate from above equation (5) modifies as,

$$C_{ij} = \frac{B}{E'}\frac{\partial^2}{\partial P_i \partial P_j}\int_0^{a_1}(K_{I1} + K_{I2})^2 da \tag{6}$$

Putting $\xi = (a/W), d\xi = \dfrac{da}{W}$

We get $da = Wd\xi$ and when a=0 $\xi = 0$;

$a = a_1$ $\xi = (a_1/W) = \xi_1$

From the above condition, equation (6) converts to

$$C_{ij} = \frac{BW}{E'}\frac{\partial^2}{\partial P_i \partial P_j}\int_0^{\xi_1}(K_{I1} + K_{I2})^2 d\xi \tag{7}$$

From the equation (7) calculating C_{11}, C_{12}=(C_{21}) and C_{22} we get

$$C_{11} = \frac{BW}{E'}\int_0^{\xi_1}\frac{\pi a}{B^2 W^2}2(F_1(\xi))^2 d\xi$$

$$= \frac{2\pi}{BE'}\int_0^{\xi_1}\xi(F_1(\xi))^2 d\xi \tag{8}$$

$$C_{12} = C_{21} = \frac{12\pi}{E'BW}\int_0^{\xi_1}\xi F_1(\xi)F_2(\xi)d\xi \tag{9}$$

$$C_{22} = \frac{72\pi}{E'BW^2}\int_0^{\xi_1}\xi F_2(\xi)F_2(\xi)d\xi \tag{10}$$

Converting the influence coefficient into dimensionless form

$$\bar{C}_{11} = C_{11}\frac{BE'}{2\pi} \tag{11}$$

$$\bar{C}_{12} = C_{12}\frac{E^{'}BW}{12\pi} = \bar{C}_{21} \tag{12}$$

$$\bar{C}_{22} = C_{22}\frac{E^{'}BW^{2}}{72\pi} \tag{13}$$

The local stiffness matrix can be obtained by taking the inversion of compliance matrix i.e.

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^{-1}$$

3.2 Governing Equations for Vibration Mode of the Cracked Beam

The cantilever beam as mentioned is being considered for free vibration analysis. Taking $U_1(x,t)$ and $U_2(x,t)$ as longitudinal vibrations for the sections before and after the crack and $Y_1(x,t)$,$Y_2(x,t)$ are the bending vibrations for the same sections, The normal function for the system can be defined as;

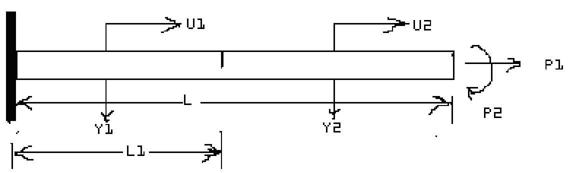

Fig. 1: Cracked Beam Model.

$$\bar{u}_1(\bar{x}) = A_1\cos(\bar{K}_u\bar{x}) + A_2\sin(\bar{K}_u\bar{x}) \tag{14a}$$

$$\bar{u}_2(\bar{x}) = A_3\cos(\bar{K}_u\bar{x}) + A_4\sin(\bar{K}_u\bar{x}) \tag{14b}$$

$$\begin{aligned}\bar{y}_1(\bar{x}) &= A_5\cosh(\bar{K}_y\bar{x}) + A_6\sinh(\bar{K}_y\bar{x}) \\ &+A_7\cos(\bar{K}_y\bar{x}) + A_8\sin(\bar{K}_y\bar{x})\end{aligned} \tag{14c}$$

$$\begin{aligned}\bar{y}_2(\bar{x}) &= A_9\cosh(\bar{K}_y\bar{x}) + A_{10}\sinh(\bar{K}_y\bar{x}) \\ &+A_{11}\cos(\bar{K}_y\bar{x}) + A_{12}\sin(\bar{K}_y\bar{x})\end{aligned} \tag{14d}$$

Where, $\bar{x} = \frac{x}{L}, \bar{u} = \frac{u}{L}, \bar{y} = \frac{y}{L}, \bar{t} = \frac{t}{L}, \beta = \frac{L_1}{L}$

$$\bar{K}_u = \frac{\omega L}{C_u}, C_u = (\frac{E}{\rho})^{0.5}, \bar{K}_y = (\frac{\omega L^2}{C_y})^{0.5}, C_y = (\frac{EI}{\mu})^{0.5}$$

$\mu = A\rho$

ω = Natural Circular Frequency

A = Shaft Cross-Section

ρ = Mass Density of the material

E =Young's Modulus of Elasticity

A_i, i = 1,12 constants are to be determined, these constants will be determined by boundary conditions.

The boundary conditions of the cantilever beam in consideration are:

$$\bar{u}_1(0) = 0 \tag{15}$$

$$\bar{y}_1(0) = 0 \tag{16}$$

$$\bar{y}_1^{'}(0) = 0 \tag{17}$$

$$\bar{u}_2^{'}(1) = 0 \tag{18}$$

$$\bar{y}_2^{''}(1) = 0 \tag{19}$$

$$\bar{y}_2^{'''}(1) = 0 \tag{20}$$

At the Cracked Section:

$$\bar{u}_1(\beta) = \bar{u}_2(\beta) \tag{21}$$

$$\bar{y}_1(\beta) = \bar{y}_2(\beta) \tag{22}$$

$$\bar{y}_1^{''}(\beta) = \bar{y}_2^{''}(\beta) \tag{23}$$

$$\bar{y}_1^{'''}(\beta) = \bar{y}_2^{'''}(\beta) \tag{24}$$

Also at cracked section we have

$$\begin{aligned}AE\frac{du_1(L_1)}{dx} &= K_{11}(u_2(L_1) - u_1(L_1)) \\ &+K_{22}(\frac{dy_2(L_1)}{dx} - \frac{dy_1(L_1)}{dx})\end{aligned}$$

Multiplying both sides by $AE/LK_{11}K_{12}$ & simplifying we get,

$$\begin{aligned}M_1M_2\bar{u}_1^{'}(\beta) &= M_2(\bar{u}_2(\beta) - \bar{u}_1(\beta)) \\ &+M_1(\bar{y}_2^{'}(\beta) - \bar{y}_1^{'}(\beta))\end{aligned} \tag{25}$$

Similarly,

$$\begin{aligned}AI\frac{d^2y_1(L_1)}{dx^2} &= K_{21}(u_2(L_1) - u_1(L_1)) \\ &+K_{22}(\frac{dy_2(L_1)}{dx} - \frac{dy_1(L_1)}{dx})\end{aligned}$$

Multiplying both sides by $AI/L^2K_{11}K_{12}$ & simplifying we get,

$$\begin{aligned}M_3M_4\bar{y}_1^{''}(\beta) &= M_3(\bar{u}_2(\beta) - \bar{u}_1(\beta)) \\ &+M_4(\bar{y}_2^{'}(\beta) - \bar{y}_1^{'}(\beta))\end{aligned} \tag{26}$$

Where, $M_1 = \frac{AE}{LK_{11}}, M_2 = \frac{AE}{K_{12}}, M_3 = \frac{EI}{LK_{22}}$ and

$$M_4 = \frac{EI}{L^2K_{21}}$$

The normal functions, equation (14) and boundary conditions (15) to (26) yield the characteristics equation of the system as:

$$|Q| = 0 \tag{27}$$

Q is a 12x12 matrix.

This determinant is a function of natural circular frequency ω ,the relative location of the crack β and the local stiffness matrix K which in turn is a function of relative crack depth (a_1 / W).

4 Experimental Set-up

An experimental set-up used for performing the experiments is shown in schematic diagram. A number of tests are conducted on Steel specimen (800mmx50mmx6mm) with a transverse crack for determining the natural frequencies and mode shapes for different crack depths. Experimental results of amplitude of transverse vibration at various locations along the length of the beam are recorded by positioning the vibration pick-up and tuning the vibration generator at the corresponding resonant frequencies.

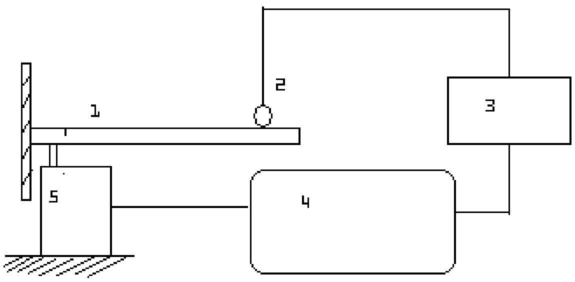

Fig. 2:Schematic diagram of experimental set-up.

1. Cracked Cantilever beam 2.Vibration pick-up
3.Vibration meter 4.Amplifier&Signal Generator
5.Electro Dynamic Exciter

5 Finite Element Analysis of Beam

The finite element analysis of cracked and uncracked beam had carried out with the help of Ansys [17] package the cracked beam was model as solid beam and it is meshed with help of tetrahedral solid elements. The cracked was taken as very fine cut. In the crack zone mesh has been properly refined. The convergent test of all the results was carried out.

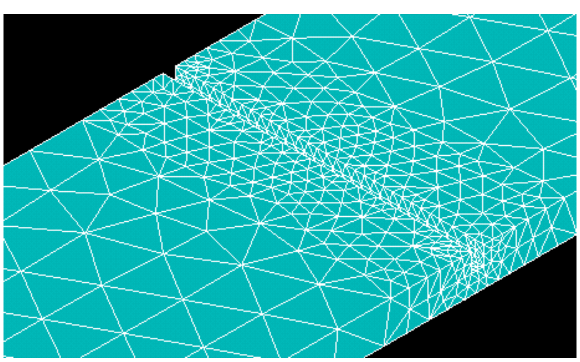

Fig. 3: Cracked Beam meshed with solid elements

7 Results

All the first natural frequency results obtain in theoretical, experimental and finite element analysis of cracked cantilever beam in hertz for different depth and different position of crack from fixed end are given below in Table 1 and Table 2.

Table-1: Analysis results of beam part-I

Depth of crack(mm)	**Position of the crack along the length from the fixed end (mm)**					
	100			**300**		
	Th (Hz)	**FEM (Hz)**	**Exp (Hz)**	**Th (Hz)**	**FEM (Hz)**	**Exp (Hz)**
0(without crack)	6.37	6.41	6.38	6.37	6.41	6.38
2	6.20	6.26	6.24	6.25	6.30	6.28
4	6.06	6.10	6.07	6.15	6.22	6.16

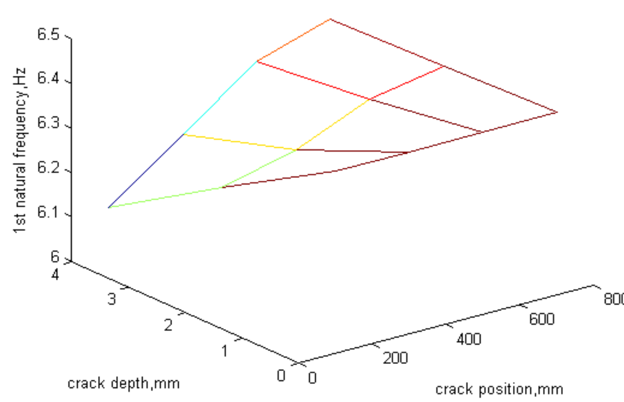

Fig. 4: 3D plot of FEM results for cracked beam

Table-2: Analysis results of beam part-II

Depth of crack(mm)	Position of the crack along the length from the fixed end (mm)					
	500			700		
	Th (Hz)	FEM (Hz)	Exp (Hz)	Th (Hz)	FEM (Hz)	Exp (Hz)
0(without crack)	6.37	6.41	6.38	6.37	6.41	6.38
2	6.30	6.37	6.31	6.34	6.40	6.36
4	6.26	6.34	6.27	6.32	6.39	6.33

6 Conclusions

Crack depth and relative crack position have got major effects on dynamic behaviors of cantilever beam. The natural frequency of a cantilever beam with transverse crack decreases with increase of crack depth. But the natural frequency shift decreases for same depth of crack as the position of the crack changes along the length from fixed end to free end of a cantilever beam.

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